A note on some best proximity point theorems proved under P-property

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Abstract. In this article, we show that some recent results on the existence of best proximity points can be obtained from the same results in fixed point theory.

Key words: Best proximity point; fixed point; weakly contractive mappings, P-property.

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1 Introduction

Let A and B be two nonempty subsets of a metric space (X, d). In this paper, we adopt the following notations and definitions.

$$D(x, B) := \inf\{d(x, y) : y \in B\}, \quad \text{for all} \quad x \in X,$$

$$A_0 := \{x \in A : d(x, y) = dist(A, B), \quad \text{for some} \quad y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = dist(A, B), \quad \text{for some} \quad x \in A\}.$$

The notion of *best proximity point* is defined as follows.

Definition 1.1. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a non-self mapping. A point $x^* \in A$ is called a best proximity point of T if $d(x^*, Tx^*) = dist(A, B)$, where*

$$dist(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Similarly, for a multivalued non-self mapping $T : A \to 2^B$, where (A, B) is a nonempty pair of subsets of a metric space (X, d), a point $x^* \in A$ is a best proximity point of T provided that $D(x^*, Tx^*) = dist(A, B)$.

Recently, the notion of P-property was introduced in [9] as follows.

Definition 1.2. *([9]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if and*

only if

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ \\ d(x_2, y_2) = dist(A, B) \end{cases} \Longrightarrow d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0$.

By using this notion, some best proximity point results were proved for various classes of non-self mappings. Here, we state some of them.

Theorem 1.3. *([9]) Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* X *such that* A_0 *is nonempty. Let* $T : A \to B$ *be a weakly contractive non-self mapping, that is,*

$$d(Tx, Ty) \le d(x, y) - \phi(d(x, y)) \ \forall x, y \in A,$$

where $\phi : [0, \infty) \to [0, \infty)$ *is a continuous and nondecreasing function such that* ϕ *is positive on* $(0, \infty), \phi(0) = 0$ *and* $\lim_{t\to\infty} \phi(t) = \infty$. *Assume that the pair* (A, B) *has the P-property and* $T(A_0) \subseteq B_0$. *Then* T *has a unique best proximity point.*

Theorem 1.4. *([1]) Let* (A, B) *be a pair of nonempty closed subsets of a Banach space* X *such that* A *is compact and* A_0 *is nonempty. Let* $T : A \to B$ *be a nonexpansive mapping, that is*

$$\|Tx - Ty\| \le \|x - y\| \ \forall x, y \in A.$$

Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then T has a best proximity point.

Theorem 1.5. *([8]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a Meir-Keeler non-self mapping, that is, for all $x, y \in A$ and for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) \leq \varepsilon.$$

Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.

Theorem 1.6. *([2]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) satisfies the P-property. Let $T : A \to 2^B$ be a multivalued contraction non-self mapping, that is,*

$$H(Tx, Ty) \leq \alpha d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x, y \in A$. If Tx is bounded and closed in B for all $x \in A$, and Tx_0 is included in B_0 for each $x_0 \in A_0$, then T has a best proximity point in A.

2 Main Result

In this section, we show that the existence of a best proximity point in the main theorems of [1, 2, 8, 9], can be obtained from the existence of the fixed point for a self-map. We begin our argument with the following lemmas.

Lemma 2.1. *([4]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and (A, B) has the P-property. Then (A_0, B_0) is a closed pair of subsets of X.*

Lemma 2.2. *Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d) such that A_0 is nonempty. Assume that the pair (A, B) has the P-property. Then there exists a bijective isometry $g : A_0 \to B_0$ such that $d(x, gx) = dist(A, B)$.*

Proof. Let $x \in A_0$, then there exists an element $y \in B_0$ such that

$$d(x, y) = dist(A, B).$$

Assume that there exists another point $\acute{y} \in B_0$ such that

$$d(x, \acute{y}) = dist(A, B).$$

By the fact that (A, B) has the P-property, we conclude that $y = \acute{y}$. Consider the non-self mapping $g : A_0 \to B_0$ such that $d(x, gx) = dist(A, B)$. Clearly, g is well defined. Moreover, g is an isometry. Indeed, if $x_1, x_2 \in A_0$ then

$$d(x_1, gx_1) = dist(A, B) \quad \& \quad d(x_2, gx_2) = dist(A, B).$$

Again, since (A, B) has the P-property,

$$d(x_1, x_2) = d(gx_1, gx_2),$$

that is, g is an isometry. $\square$

Here, we prove that the existence and uniqueness of the best proximity point in Theorem 1.3 is a sample result of the existence of fixed point for a weakly contractive self-mapping.

Theorem 2.3. *Let (A, B) be a pair of nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping. Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.*

Proof. Consider the bijective isometry $g : A_0 \to B_0$ as in Lemma 2.2. Since $T(A_0) \subseteq B_0$, for the self-mapping $g^{-1}T : A_0 \to A_0$ we have

$$d(g^{-1}(Tx), g^{-1}(Ty)) = d(Tx, Ty) \leq \varphi(d(x, y)),$$

for all $x, y \in A_0$ which implies that the self-mapping $g^{-1}T$ is weakly contractive. Note that A_0 is closed by Lemma 2.1. Thus, $g^{-1}T$ has a unique fixed point ([7]). Suppose that $x^* \in A_0$ is a unique fixed point of the self-mapping $g^{-1}T$, that is, $g^{-1}T(x^*) = x^*$. So, $Tx^* = gx^*$ and then

$$d(x^*, Tx^*) = d(x^*, gx^*) = dist(A, B),$$

from which it follows that $x^* \in A_0$ is a unique best proximity point of the non-self weakly contractive mapping T. $\square$

Remark 2.1. By a similar argument, using the fact that every nonexpansive self-mapping defined on a nonempty compact and convex subset of a Banach space has a fixed point, we conclude Theorem 1.4. Also, existence and uniqueness of the best proximity point for Meir-Keeler non-self mapping T follows from the Meir-Keeler's fixed point theorem ([5]). Finally, in Theorem 1.5, the Nadler's fixed point theorem ([6]), ensures the existence of a best proximity point for multivalued non-self T.

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